

# Economic inequality and mobility for stochastic models with multiplicative noise ‡

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**Abstract.** In this article, we discuss a dynamical stochastic model that represents the time evolution of income distribution of a population, where the dynamics develop from an interplay of multiple economic exchanges in the presence of multiplicative noise. The model remit stretches beyond the conventional framework of a Langevin-type kinetic equation in that our model dynamics is self-consistently constrained by dynamical conservation laws emerging from population and wealth conservation. This model is numerically solved and analyzed to interpret the inequality of income as a function of relevant dynamical parameters like the *mobility*  $M$  and the *total income*  $\mu$ . In our model, inequality is quantified by the *Gini index*  $G$ . In particular, correlations between any two of the mobility index  $M$  and/or the total income  $\mu$  with the Gini index  $G$  are investigated and compared with the analogous correlations resulting from an equivalent additive noise model. Our findings highlight the importance of a multiplicative noise based economic modeling structure in the analysis of inequality. The model also depicts the nature of correlation between mobility and total income of a population from the perspective of inequality measure.

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## 1. Introduction

Various approaches inspired by a combination of statistical physics and kinetic theory have been proposed in recent years for the description of economic exchanges and market societies, see for example [1, 2, 3, 4, 5, 6, 7]. In these approaches, individuals trading with each other are identified as particles or gas molecules which undergo collisions. Methods and tools based on physics have proved useful also in such socio-economic contexts to investigate the emergence of macroscopic features from a whole of microscopic interactions. With this perspective, some mathematically founded market economy models, characterized by the ability to also incorporate taxation and redistribution processes, have been proposed and studied in [8, 9]. In these papers, society is equated to a system composed by a large number of heterogeneous individuals who exchange money through binary and other nonlinear interactions and are divided into a finite number  $n$  of income classes. The models are expressed by a system of  $n$  nonlinear ordinary differential equations of the kinetic-discretized Boltzmann type, involving transition probabilities relative to the jumps of individuals from a class to another. The specification of these probabilities and of the parameters which define the trading rules, including the tax rates pertaining to different income classes and other properties of the system, determines the dynamics. Collective features like the income profile and related indicators like the Gini index - here, a measure of economic inequality - result from the interplay of a range of such interactions. Due to the presence of the mentioned transition probabilities, the process is stochastic [10] but the differential equations governing the evolution of the fraction of individuals in the classes are deterministic.

In real world, however, the time evolution of an economic system is governed not only by fixed rules and parameters: it is subject to the effects of unpredictable perturbing factors as well. To consider the influence of these factors, we recently introduced a Langevin-type kinetic model [11], incorporating an Ito-type additive noise term into the set of dynamical equations. Several numerical simulations provided evidence of the persistence of patterns already established in the deterministic problem [12], also in agreement with previously explored empirical results [13, 14, 15]: in particular, they exhibited a negative correlation between economic inequality and social mobility. With reference to the case without income conservation they reported a positive correlation between the Gini index and the total income. We regard this as a sign of reliability of the models. The noise additivity is a perceived drawback though, as it does not prevent uncontrollably large fluctuations from being compared to class populations, which is unrealistic.

The goal of this paper is to overcome this limitation by considering instead a multiplicative noise term. This requires a more subtle procedure than that proposed in [11]. Attention is then focused on the sign of the correlations between income inequality, mobility and total income under different conditions as described in Section 3 below.

The paper is organized as follows. In the next section, we introduce the structure of the Langevin-type kinetic model. Different choices for the construction of the noise

term of this structure allow to formulate different models. Here, in particular, we define two of them: one, in the first subsection, for which only conservation of the total population holds true, and another, in a second subsection, for which both conservation of total population and total income hold true. The features of the evolution in time of the solutions of these two models are discussed in Section 3. If the total income  $\mu$  is constant in time and not too large, the correlations between the Gini index and an indicator quantifying social mobility is negative. The same negativity was obtained in [11] in the presence of additive noise, the only difference being the absence of any restriction on the values of  $\mu$ . When income conservation does not hold true, the sign of the correlation between the total income and the Gini index can either be positive or negative, depending on the magnitude of  $\mu$ , when the noise is multiplicative (the case in the present paper). On the other hand, the correlation is positive when the noise is additive (the case in [11]). The conclusion summarizes these facts and some directions for future research.

## 2. From a deterministic to a Langevin-type kinetic model

A simple model describing monetary exchanges between pairs of individuals in a society divided into  $n$  income classes can be formulated through a system of differential equations of the form

$$\frac{dx_i}{dt}(t) = \sum_{h,k=1}^n C_{hk}^i x_h(t) x_k(t) - \sum_{h,k=1}^n C_{ik}^h x_i(t) x_k(t), \quad i = 1, \dots, n. \quad (1)$$

Here,  $x_i(t)$  denotes the fraction of individuals which at time  $t$  belong to the  $i$ -th class and the constant coefficients  $C_{hk}^i \in [0, 1]$ , such that  $\sum_{i=1}^n C_{hk}^i = 1$  for any fixed  $h$  and  $k$ , express the probability that an individual of the  $h$ -th class will belong to the  $i$ -th class after a direct interaction with an individual of the  $k$ -th class. The expression for these coefficients, valid for the case in which the average incomes are given by

$$r_j = j \cdot \Delta r, \quad (2)$$

with  $\Delta r > 0$ , first derived in [16] and then used e.g. also in [6, 8, 9], reads when written in compact form:

$$\begin{aligned} C_{hk}^i = & \frac{S}{\Delta r} \delta_{hi} \left[ \frac{\Delta r}{S} - (1 - \delta_{in})(1 - \delta_{k1}) p_{ki} - (1 - \delta_{i1})(1 - \delta_{kn}) p_{ik} \right] \\ & + \frac{S}{\Delta r} \left[ \delta_{h,i+1}(1 - \delta_{kn}) p_{i+1,k} + \delta_{h,i-1}(1 - \delta_{k1}) p_{k,i-1} \right] \end{aligned} \quad (3)$$

with  $h, k, i = 1, \dots, n$ . Here,  $S \ll \Delta r$  denotes a unit of money,  $\delta_{hk}$  denotes the Kronecker's delta, and  $p_{hk}$  expresses the probability that in an encounter between an individual of the  $h$ -th class and one of the  $k$ -th class, the one who pays is the former one. We take here

$$p_{hk} = \frac{1}{4n} \min\{h, k\} (1 - \delta_{hk}) (1 - \delta_{1k}) (1 - \delta_{1h}) (1 - \delta_{nh}) (1 - \delta_{nk})$$

$$\begin{aligned}
& + \frac{h}{2n} \delta_{hk} (1 - \delta_{1k}) (1 - \delta_{nk}) + \frac{k}{2n} \delta_{nh} (1 - \delta_{nk}) (1 - \delta_{1k}) \\
& + \frac{1}{2n} \delta_{1k} (1 - \delta_{1h}) (1 - \delta_{nh}) + \frac{1}{2n} \delta_{hn} \delta_{k1},
\end{aligned} \tag{4}$$

and extend the values of  $p_{hk}$  in (4) to allow the indices  $h$  and  $k$  to go from 0 to  $n + 1$  through the definitions  $p_{n+1,k} = 0$  for any  $k$ , and  $p_{k,0} = 0$  for any  $k$ .

We emphasize that the choice of the coefficients (3) is forced, if conservation of total income has to hold true for all  $t \geq 0$  once it holds true for  $t = 0$  (see the proof of Theorem 4.2 in [16]). In contrast, there is a certain degree of arbitrariness in the choice of  $p_{hk}$ . The specific formula (4) corresponds to a choice made in [9], suggested by the observation that usually poor people pay and receive less than rich people. Indeed, the  $p_{hk}$  in (4) with indices  $h$  and  $k$  different from 1 and  $n$  are equal to  $\frac{1}{4} \frac{\min\{r_h, r_k\}}{r_n}$  (and those with  $h = k$  are equal to  $\frac{1}{2} \frac{\min\{r_h, r_k\}}{r_n}$ ). The special treatment of the coefficients with indices  $h, k = 1$  or  $n$  is due to the impossibility of moving from the first class to a poorer one and from the  $n$ -th class to a richer one.

A Langevin-type kinetic model [18] can now be constructed as a system of stochastic equations of the form

$$dx_i = D_i^{(1)}(x)dt + \sum_{j=1}^n D_{ij}^{(2)}(x)\xi_j\sqrt{\Gamma}dt, \quad i = 1, \dots, n, \tag{5}$$

in which the first term on the r.h.s. in Eq. (5) represents the “deterministic” contribution and the second term corresponds to noise. The interpretation of equation (5) is as follows. The first term takes into account direct money exchanges, ruled by norms, and behavioral attitudes which are the same for individuals belonging to the same class. The second term represents uncertainties randomly occurring, which also affect the change in the population distribution.

In the following we take the operator  $D_i^{(1)}$  as in (1),

$$D_i^{(1)}(x) = \sum_{h,k} C_{hk}^i x_h x_k - \sum_{h,k} C_{ik}^h x_i x_k,$$

i.e. we take  $D_i^{(1)}$  to mimic that component of the models in [6, 8] which just describes the direct monetary exchanges without taxation and redistribution. As for the stochastic part, the  $\xi_i$  denote  $n$  independent Gaussian stochastic variables and  $\Gamma$  denotes the noise amplitude. The form of the operator  $D_{ij}^{(2)}$  depends on the conservation requirements to which we want the model to obey.

### 2.1. Multiplicative noise with conserved total population

Enforcing total population conservation, we must get

$$\sum_{i,j} D_{ij}^{(2)}(x)\xi_j = 0, \tag{6}$$

for any choice of  $\{\xi_j\}$ . A way to fulfill condition (6) together with a proportionality condition between the random variations in the class populations and the population themselves is to define, starting from the random  $\xi_i$ , new variations  $\xi'_i = x_i \xi_i - x_i \sum_k x_k \xi_k$ , or in matrix form

$$\xi'_i = \sum_j D_{ij}^{(2)}(x) \xi_j,$$

with

$$D_{ij[\text{pop-const}]}^{(2)}(x) = \begin{cases} x_i (1 - x_i), & \text{if } i = j \\ -x_i x_j, & \text{if } i \neq j. \end{cases} \quad (7)$$

The formula (7) provides an operator  $D_{ij[\text{pop-const}]}^{(2)}$  which allows to construct, starting from random variables, a multiplicative noise term compatible with the conservation of the total population (“pop-const”). Incidentally, we observe that in the following, as in [6, 8, 12], we normalize the total population to 1. On the contrary, we emphasize that allowing a variation of the total income related to stochastic noise amounts to consider for example a society which also interacts with the “external world”: capital inflow or outflow could occur due to import-export of goods, incoming-outgoing of tourism, investment and stock trading.

## 2.2. Multiplicative noise with conserved total population and income

Alternatively, we may consider a closed system for which we require conservation of the total income  $\mu = \sum_i r_i x_i \xi_i$ . We then point out that from now on we restrict attention on values of  $\mu$  satisfying

$$r_1 < \mu < r_n. \quad (8)$$

Thanks to the inequalities (8) the possible occurrence can be excluded of situations in which all individuals belong to the poorest or to the richest income class. Similar odd cases are not of interest if one wants to deal with realistic situations. In other words, taking  $\mu$  as in (8) does not represent a strong assumption.

In addition to (6), a further condition has now to be imposed, i.e.

$$\sum_{i,j} r_i D_{ij}^{(2)}(x) \xi_j = 0 \quad (9)$$

for any choice of  $\{\xi_j\}$ . In order to construct a diffusion matrix satisfying both (6) and (9), we begin by proving the following proposition.

**Proposition 1.** Given a vector  $x = (x_1, \dots, x_n)$  with  $x_i > 0$  for all  $i$ , and  $n$  positive constants  $r_i$ , from any vector  $\eta_0 = (\eta_{01}, \dots, \eta_{0n})$  with  $|\eta_{0i}| \leq 1$  for all  $i$ , a new vector  $\bar{\eta} = (\bar{\eta}_1, \dots, \bar{\eta}_n)$  may be obtained, which satisfies the estimates

$$|\bar{\eta}_i| \leq x_i \quad \text{for } i = 1, \dots, n$$

§ Notice that, due to the normalization to 1 of the population, the total income coincides with the average income of the population itself.

and the two conditions

$$\sum_i \bar{\eta}_i = 0, \quad \text{and} \quad \sum_i r_i \bar{\eta}_i = 0. \quad (10)$$

*Proof:* We begin by associating to  $\eta_0$  a vector

$$\eta = (\eta_1, \dots, \eta_n) = \left( \frac{x_1 \eta_{0,1}}{C}, \dots, \frac{x_n \eta_{0,n}}{C} \right), \quad (11)$$

where  $C \geq 1$  is a constant to be determined in the following. We want then to transform the vector  $\eta$  to a perturbed vector  $\bar{\eta} = \eta + A\eta$ , with components

$$\bar{\eta}_i = \eta_i + \sum_{j=1}^n a_{ij} \eta_j \quad \text{for } i = 1, \dots, n \quad (12)$$

satisfying the conservation conditions (10). Inserting (12) in (10), we find (keeping also the arbitrariness of  $\eta_{0,i}$  into account) that the conditions (10) become

$$1 + \sum_{j=1}^n a_{ji} = 0, \quad \text{and} \quad r_i + \sum_{j=1}^n a_{ji} r_j = 0$$

for  $i = 1, \dots, n$ . If we choose the matrix  $A$  in the set of tridiagonal matrices<sup>||</sup>, these conditions read<sup>¶</sup>

$$1 + \sum_{j=i-1}^{i+1} a_{ji} = 0, \quad \text{and} \quad r_i + \sum_{j=i-1}^{i+1} a_{ji} r_j = 0 \quad (13)$$

for  $i = 1, \dots, n$ . The formulas (13) express  $2n$  constraints which the  $3n - 2$  elements  $a_{ij}$  of the matrix  $A$  have to satisfy<sup>+</sup>. We then minimize the function of the  $3n - 2$  variables  $a_{ji}$ ,

$$f = \sum_{i=1}^n \sum_{j=i-1}^{i+1} a_{ji}^2$$

subject to the  $2n$  constraints (13). To this end, we introduce Lagrange multipliers  $\lambda_i$  and  $\mu_i$  for  $i = 1, \dots, n$ , and consider the Lagrangian

$$L = \sum_{i=1}^n \sum_{j=i-1}^{i+1} a_{ji}^2 + \sum_{i=1}^n \lambda_i \left( 1 + \sum_{j=i-1}^{i+1} a_{ji} \right) + \sum_{i=1}^n \mu_i \left( r_i + \sum_{j=i-1}^{i+1} a_{ji} r_j \right).$$

The search for critical points of  $L$  (as a function of the variables  $a_{ji}$ ,  $\lambda_i$  and  $\mu_i$ ) yields in particular, after straightforward calculations,

$$a_{ji} = \frac{N_i r_i r_j + T_i - R_i r_i - R_i r_j}{R_i^2 - N_i T_i}, \quad (14)$$

<sup>||</sup> The reason is that with this choice the variation of the  $i$ -th component when passing from  $\eta$  to  $\bar{\eta}$  only involves  $\eta_{i-1}$ ,  $\eta_i$ , and  $\eta_{i+1}$ .

<sup>¶</sup> Here and henceforth only indexed terms with meaningful indices are to be considered present. For example, if  $i = 1$ , one has  $\sum_{j=i-1}^{i+1} a_{ji} = a_{11} + a_{21}$ .

<sup>+</sup> It is natural to assume  $n \geq 3$  here.

for  $i = 1, \dots, n, j = i - 1, i, i + 1$  (the remaining  $a_{ji}$  being equal to zero), where

$$N_1 = 2, \quad N_i = 3 \quad \text{for } i = 2, \dots, n - 1, \quad N_n = 2,$$

and

$$R_i = \sum_{k=i-1}^{i+1} r_k \quad \text{and} \quad T_i = \sum_{k=i-1}^{i+1} r_k^2.$$

In view of the linearity of  $r_j$  in  $j$  as formulated in Eq. (2), it can be easily seen that the matrix  $A$  with elements as in (14) takes the form

$$A = \begin{bmatrix} -1 & -1/3 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/3 & -1/3 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/3 & -1/3 & -1/3 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/3 & -1/3 & -1/3 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & -1/3 & -1/3 & -1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & -1/3 & -1/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & -1/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & -1/3 & -1 \end{bmatrix}.$$

We observe now that applying the transformation (12) with the matrix  $A$  just found, we get

$$\frac{\bar{\eta}_i}{x_i} = \frac{\eta_i}{x_i} + \frac{\sum_{j=i-1}^{i+1} a_{ij} \eta_j}{x_i} \quad \text{for } i = 1, \dots, n, \quad (15)$$

namely,

$$\begin{aligned} \frac{\bar{\eta}_1}{x_1} &= -\frac{1}{3C} \frac{x_2}{x_1} \eta_{0,2}, \\ \frac{\bar{\eta}_2}{x_2} &= \frac{2}{3C} \eta_{0,2} - \frac{1}{3C} \frac{x_3}{x_2} \eta_{0,3}, \\ \frac{\bar{\eta}_i}{x_i} &= \frac{2}{3C} \eta_{0,i} - \frac{1}{3C} \frac{x_{i-1}}{x_i} \eta_{0,i-1} - \frac{1}{3C} \frac{x_{i+1}}{x_i} \eta_{0,i+1}, \quad \text{for } i = 3, \dots, n-2, \\ \frac{\bar{\eta}_{n-1}}{x_{n-1}} &= \frac{2}{3C} \eta_{0,n-1} - \frac{1}{3C} \frac{x_{n-2}}{x_{n-1}} \eta_{0,n-2}, \\ \frac{\bar{\eta}_n}{x_n} &= -\frac{1}{3C} \frac{x_{n-1}}{x_n} \eta_{0,n-1}. \end{aligned}$$

For the choice of the constant  $C$  appearing here and in (11), we first calculate

$$M_{minus} = \max_{i=2, \dots, n} \left\{ \frac{x_i}{x_{i-1}} \right\} \quad \text{and} \quad M_{plus} = \max_{i=1, \dots, n-1} \left\{ \frac{x_i}{x_{i+1}} \right\}, \quad (16)$$

and set

$$\Omega = \max \left\{ 1, M_{minus}, M_{plus} \right\}. \quad (17)$$

Then, we fix the constant  $C$  in (11) to be equal to  $\frac{4}{3}\Omega$ . Hence, according to (11), we associate to a randomly chosen vector  $\eta_0$  the vector

$$\eta = (\eta_1, \dots, \eta_n) = \left( \frac{3}{4} \frac{x_1 \eta_{0,1}}{\Omega}, \dots, \frac{3}{4} \frac{x_n \eta_{0,n}}{\Omega} \right). \quad (18)$$

Now, applying to  $\eta$  the transformation (12) with the  $a_{ij}'$ s as in (14), we get

$$\begin{aligned} \frac{\bar{\eta}_1}{x_1} &= \frac{3}{4} \left( -\frac{1}{3} \frac{x_2}{x_1} \frac{1}{\Omega} \eta_{0,2} \right), \\ \frac{\bar{\eta}_2}{x_2} &= \frac{3}{4} \left( \frac{2}{3} \frac{1}{\Omega} \eta_{0,2} - \frac{1}{3} \frac{x_3}{x_2} \frac{1}{\Omega} \eta_{0,3} \right), \\ \frac{\bar{\eta}_i}{x_i} &= \frac{3}{4} \left( \frac{2}{3} \frac{1}{\Omega} \eta_{0,i} - \frac{1}{3} \frac{x_{i-1}}{x_i} \frac{1}{\Omega} \eta_{0,i-1} - \frac{1}{3} \frac{x_{i+1}}{x_i} \frac{1}{\Omega} \eta_{0,i+1} \right), \quad \text{for } i = 3, \dots, n-2, \\ \frac{\bar{\eta}_{n-1}}{x_{n-1}} &= \frac{3}{4} \left( \frac{2}{3} \frac{1}{\Omega} \eta_{0,n-1} - \frac{1}{3} \frac{x_{n-2}}{x_{n-1}} \frac{1}{\Omega} \eta_{0,n-2} \right), \\ \frac{\bar{\eta}_n}{x_n} &= \frac{3}{4} \left( -\frac{1}{3} \frac{x_{n-1}}{x_n} \frac{1}{\Omega} \eta_{0,n-1} \right), \end{aligned}$$

which in turn implies

$$\begin{aligned} \left| \frac{\bar{\eta}_1}{x_1} \right| &\leq \frac{1}{4} |\eta_{0,2}| \leq 1, \\ \left| \frac{\bar{\eta}_2}{x_2} \right| &\leq \frac{2}{4} |\eta_{0,2}| + \frac{1}{4} |\eta_{0,3}| \leq 1, \\ \left| \frac{\bar{\eta}_i}{x_i} \right| &\leq \frac{2}{4} |\eta_{0,i}| + \frac{1}{4} |\eta_{0,i-1}| + \frac{1}{4} |\eta_{0,i+1}| \leq 1, \quad \text{for } i = 3, \dots, n-2, \\ \left| \frac{\bar{\eta}_{n-1}}{x_{n-1}} \right| &\leq \frac{2}{4} |\eta_{0,n-1}| + \frac{1}{4} |\eta_{0,n-2}| \leq 1, \\ \left| \frac{\bar{\eta}_n}{x_n} \right| &\leq \frac{1}{4} |\eta_{0,n-1}| \leq 1. \end{aligned}$$

In conclusion, the vector  $\bar{\eta}$  satisfies the conservation conditions given in Eq. (10) as well as the estimates  $|\bar{\eta}_i| \leq x_i$  for  $i = 1, \dots, n$ .  $\square$

In order to construct from the stochastic variable  $\xi$  a multiplicative noise term satisfying conservation of population and income, one can discretize time and repeatedly iterate, as illustrated below, the procedure of Proposition 1. We emphasize that a warning as discussed in the next lines is in order here.

At each step, say at each time  $t_k$  with  $k = 0, 1, 2, \dots$ , a vector  $\xi$  is picked whose components  $\xi_i$  for  $i = 1, \dots, n$  are Gaussian random numbers ranging from  $-1$  to  $1$ . Here,  $\xi$  plays the role of  $\eta_0$  in Proposition 1. The vector  $x = (x_1, \dots, x_n)$  of Proposition 1 is given at the beginning of the process, i.e. at time  $t_0$ , by a stationary distribution  $x_{eq}$  (reached in the long run) of the “deterministic” system (1), whereas at subsequent steps, i.e. at time  $t_k$  with  $k = 1, 2, \dots$ , it is given by the solution  $x(t_k)$  of the system (5), or of the system (1), according to the criterion described next. There are two possibilities: either  $x_i > 0$  for all  $i = 1, \dots, n$  or there exists at least an index  $i^*$  such



that  $x_{i^*}$  vanishes.\* A control loop in the algorithm checks which of the two possibilities holds true. Accordingly, the procedure to be applied is as follows.

- (i) If at time  $t_k$  it is  $x_i > 0$  for all  $i = 1, \dots, n$ , one calculates  $\Omega$  according to Eq.s (16) and (17) and then defines, by applying the formula (18) with this value of  $\Omega$ , an “intermediate” vector  $\eta$ . Then, one applies to  $\eta$  the transformation (12) with the  $a_{ij}$ ’s as in (14). In this way one obtains, as Proposition 1 shows, a vector whose components are proportional to the classes populations and which, when inserted in the equation (5), guarantees both population and total income conservation (“pop-inc-const”). This vector can be denoted by

$$D_{[pop-inc-const]}^{(2)}(x)\xi. \quad (19)$$

Numerical solutions of (5) can be found by calculating (19), inserting the noise term

$$D_{ij[pop-inc-const]}^{(2)}(x)\xi_j\sqrt{\Gamma dt}$$

into the equation (5) and getting the corresponding solution  $x(t_{k+1})$ . If  $x_i(t_{k+1}) > 0$  for all  $i = 1, \dots, n$  and all  $k \in \mathbf{N}$ , one repeats all this over and over again.

- (ii) If for some integer  $k$  and some index  $i^*$ , the component  $x_{i^*}(t_k)$  vanishes, i.e. denoting  $t_k = t^*$  one has  $x_{i^*}(t^*) = 0$ , then one lets only the system (1) evolve, without adding any noise up to when  $x_i > 0$  for all  $i = 1, \dots, n$ . From then on, the algorithm described in 1 has to be applied again. To give an insight as to why the re-establishment of the situation with all  $x_i > 0$  is to be expected, we argue as follows.

First of all, we want to exclude the cases (both of which are equilibria for the system (1)), for which all individuals belong to the poorest class or to the richest class. Since the value of the total income with which the former case is compatible is  $\mu = r_1$  whereas for the latter case it is  $\mu = r_n$ , the assumption (8) guarantees that these cases cannot occur, thereby assuring “moderate income” remit.

We then observe that exploiting the fact that  $x_{i^*}(t^*) = 0$  one gets from (1),

$$\frac{dx_{i^*}}{dt}(t^*) = \sum_{h \neq i^*} \sum_{k \neq i^*} C_{hk}^{i^*} x_h(t^*) x_k(t^*) \geq 0. \quad (20)$$

It is of course possible that other  $x_i$  in addition to  $x_{i^*}$  vanish at time  $t^*$ . Then, let  $m$  be the smallest positive integer such that

$$x_{i^*-m}(t^*) \neq 0 \quad \text{or} \quad x_{i^*+m}(t^*) \neq 0 \quad (21)$$

holds true. Such a number certainly exists. Assume, without loss of generality, the second of the two inequalities (21) to hold true. The other case can be handled similarly. Now, either  $i^* + m < n$  or  $i^* + m = n$  holds true.

\* In fact it is highly improbable that the second alternative occurs. Nevertheless, we take it too into consideration.

- If  $i^* + m < n$ , observing that  $C_{ii}^{i-1} > 0$  (as also  $C_{ii}^{i+1} > 0$ ) provided  $1 < i < n$ , we conclude that  $C_{i^*+m, i^*+m}^{i^*+m-1} > 0$  and hence

$$\frac{d}{dt}x_{i^*+m-1}(t^*) \geq C_{i^*+m, i^*+m}^{i^*+m-1} x_{i^*+m}^2(t^*) > 0.$$

Consequently,  $x_{i^*+m-1}(t^* + 1) > 0$ . Iterating the procedure  $m$  times, one obtains

$$x_{i^*}(t^* + m) > 0.$$

- If  $i^* + m = n$ , we know, in view of (8), that there exists a positive integer  $p$ , satisfying  $1 \leq i^* - p$ , such that  $x_{i^*-p}(t^*) \neq 0$ . If  $i^* - p > 1$ , then, similarly as above, one notices that  $\frac{d}{dt}x_{i^*-p+1}(t^*) \geq C_{i^*-p, i^*-p}^{i^*-p+1} x_{i^*-p}^2(t^*) > 0$ , from which  $x_{i^*-p+1}(t^* + 1) > 0$  and then  $x_{i^*}(t^* + p) > 0$  follows. If  $i^* - p = 1$ , then one may exploit the fact that  $C_{1n}^2 > 0$  and  $\frac{d}{dt}x_{i^*-p+1}(t^*) \geq C_{1n}^2 x_1(t^*)x_n(t^*) > 0$  to be reconduced to the case just dealt with.

By repeating, if necessary, the procedure here illustrated, one ends up with  $x_i(t_{k+q}) > 0$  for all  $i = 1, \dots, n$ , for some  $q \in \mathbf{N}$ .

### 3. Simulation results

To investigate the stochastic processes of the two models designed in Section 2, we numerically solved the equations (5) and took the average of various quantities over a large number of stochastic realizations. Of course, no equilibria have to be expected in the present case. To draw some conclusions, we need to recall the definition - more precisely, a variant of it, suitable for the present case - of an indicator of social mobility introduced in [12]. This indicator, which expresses the collective probability of class advancement of all classes from the 2-th to the  $(n - 1)$ -th one, is given by

$$M = \frac{1}{(1 - x_1 - x_n)} \sum_{i=2}^{n-1} \sum_{k=1}^n \frac{S}{(r_{i+1} - r_i)} p_{k,i} x_k x_i.$$

We calculated the value of  $M$  in a succession of equally spaced instants  $\{t_j\}$  along the evolution in time of several solutions of Eq. (5). As well, in correspondence to the same instants, we calculated the Gini index  $G^\sharp$ .

A significant finding concerns the sign of the correlation between  $G$  and  $M$ , namely between economic inequality and social mobility. For values of the total income  $\mu$  which are not too large when total income is conserved, and which are neither too large nor too small when total income is not conserved, the statistical value of the sign of the correlation between  $G$  and  $M$  turns out to be negative. The values of  $\mu$  under

$\sharp$  This coefficient was introduced by the italian statistician Corrado Gini a century ago. It takes values in  $[0, 1]$  and it is defined as a ratio, having the numerator given by the area between the Lorenz curve of a distribution and the uniform distribution line, and the denominator given by the area of the region under the uniform distribution line.

consideration<sup>††</sup> are reasonable in a realistic perspective (see e.g. [17]) because they are compatible with a distribution of individuals in which most of the population belongs to the low-middle classes. The negativity of the correlation which we get is in agreement with a great deal of empirical data [13, 14] and provides evidence of some robustness against random perturbations of the corresponding property established for systems without noise in [12]. A few samples of correlations  $R_{GM}$  (Gini and mobility index) are given in Table 1 for the case with constant total income and in Table 2 for the case with varying total income. For our simulations, we considered the difference  $\Delta r$  between class average incomes equal to 10 and the noise amplitude  $\Gamma$  equal to 0.001. The correlations were obtained as averages of 50 realizations, each over 5000 integration steps. In these samples, three initial conditions - the same in the Tables 1 and 2 - compatible with values of the total income equal to 24.5, 27 and 29.5 respectively, are considered (Figure 1 displays the initial condition corresponding to the asymptotic stationary distribution for the system without noise (1) with  $\mu = 27$ ). And for each of these initial conditions, three different average results are reported.

We also stress here that the distributions we get after the 5000 integration steps remain in fact quite “close” to the distributions from which they evolve, which are equilibria if noise is absent. We measure the “closeness” by calculating in correspondence to each realization the difference between the average (of the 5000 values attained during evolution)  $\hat{x}_i$  of each component of the distribution and the corresponding initial value  $x_i(0)$ ; in addition, we calculate the standard deviation  $\sigma_{x_i}$  of each component  $x_i$ . We find that the differences  $\hat{x}_i - x_i(0)$  take values whose order of magnitude typically are between  $10^{-5}$  and  $10^{-7}$ , the  $\sigma_{x_i}$  take values whose order of magnitude typically oscillate between  $10^{-4}$  and  $10^{-6}$ , whereas the values of the relative standard variations  $\sigma_{x_i}/x_i$  typically are of the order of  $10^{-4}$ .

$\mu$	$R_{GM}$	$R_{GM}$	$R_{GM}$
24.5	- 0.980 $\pm$ 0.002	- 0.984 $\pm$ 0.001	- 0.983 $\pm$ 0.002
27.0	- 0.967 $\pm$ 0.003	- 0.970 $\pm$ 0.003	- 0.968 $\pm$ 0.003
29.5	- 0.913 $\pm$ 0.007	- 0.923 $\pm$ 0.008	- 0.920 $\pm$ 0.007

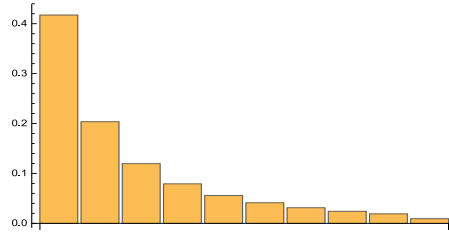
**Table 1.** Correlations  $R_{GM}$  (Gini and mobility index) computed in nine cases in which total income  $\mu$  is conserved, with noise amplitude  $\Gamma = 0.001$ . Averages of 50 realizations, each of 5000 integration steps.

A further issue which one can explore in the non conservative case is the correlation  $R_{G\mu}$  between the Gini index and the total income. A difference comes out in this respect, depending on whether the noise is additive or multiplicative: whereas the value of  $R_{G\mu}$  provided by the numerical simulations is positive in the first case, it turns out to be

<sup>††</sup>For example, if we take  $n = 10$  and fix the values of  $r_i$  for  $i = 1, \dots, n$  to be linearly growing from  $r_1 = 10$  to  $r_{10} = 100$ , values of  $\mu \leq 30$  in the conservative case and  $\mu \in [24, 30]$  in the non conservative case meet this criterion.

$\mu(0)$	$R_{GM}$	$R_{GM}$	$R_{GM}$
24.5	$- 0.150 \pm 0.061$	$- 0.204 \pm 0.056$	$- 0.220 \pm 0.062$
27.0	$- 0.276 \pm 0.064$	$- 0.475 \pm 0.051$	$- 0.450 \pm 0.052$
29.5	$- 0.610 \pm 0.044$	$- 0.611 \pm 0.034$	$- 0.605 \pm 0.047$
$\mu(0)$	$R_{G\mu}$	$R_{G\mu}$	$R_{G\mu}$
24.5	$0.096 \pm 0.061$	$0.043 \pm 0.059$	$0.045 \pm 0.063$
27.0	$- 0.068 \pm 0.067$	$- 0.271 \pm 0.059$	$- 0.239 \pm 0.058$
29.5	$- 0.465 \pm 0.052$	$- 0.443 \pm 0.043$	$- 0.466 \pm 0.054$

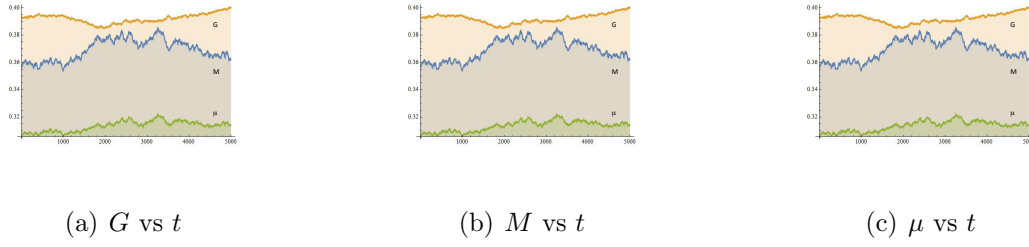
**Table 2.** Correlations  $R_{GM}$  (Gini and mobility index) and  $R_{G\mu}$  (Gini index and total income) computed in nine cases in which total income  $\mu$  is not conserved, with noise amplitude  $\Gamma = 0.001$ . Averages of 50 realizations, each of 5000 integration steps.



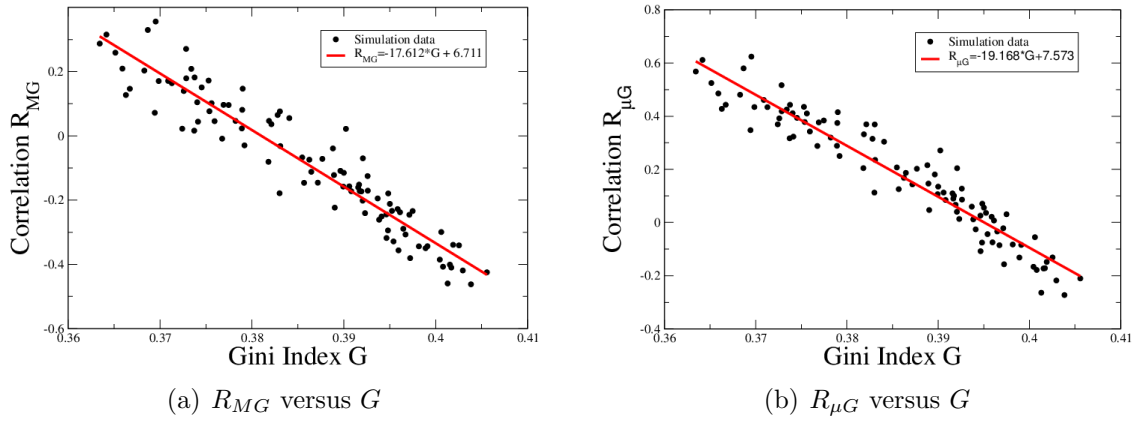
**Figure 1.** The asymptotic stationary solution of the “deterministic” system with constant total income  $\mu = 27$ .

sometimes negative and sometimes positive in the second one, depending on the value of the initial total income  $\mu$ . An intuitive argument for a possible explanation of the positive sign in the additive case is as follows: in the presence of additive noise the variations in the rich classes are typically much larger (with respect to those in the low and middle classes) than when the noise is multiplicative. This causes larger variations in the total income. Since increases of  $\mu$  mainly affect the richer classes, this brings about an increase of inequality, i.e. of  $G$ . Yet, we do not have an explanation for the behavior of the correlation  $R_{G\mu}$  in the multiplicative case. It has also to be noticed that the values reported in the Table 2 evidenciate a great variability (and possibly, even no meaningfulness) of  $R_{G\mu}$ , when the total income is not fixed. We notice however that a strong positive correlation  $R_{M\mu}$  between mobility and total income comes out of the realizations. A few samples of that are reported in the Table 3. Also, from the three panels in Figure 2 displaying time series of  $G$ ,  $M$  and  $\mu$  the negativity of the correlation between  $G$  and  $M$  and the positivity of the correlation between  $M$  and  $\mu$  is clearly visible.

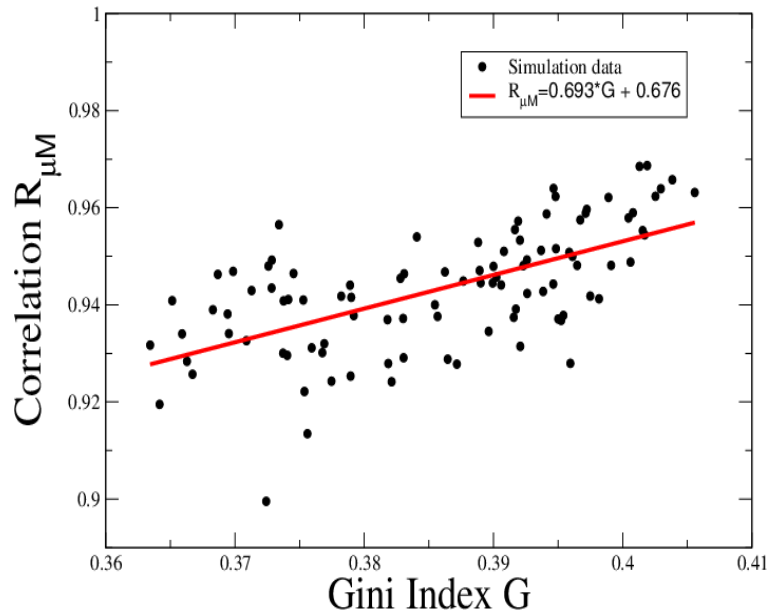
As can be seen in the Tables 2 and 3 the correlations between  $G$ ,  $M$  and  $\mu$  depend on the value of  $\mu$ . Since the values of  $\mu$  and  $G$  at equilibrium are mutually related, the correlations depend on  $G$ . In order to further check this dependence, we ran simulations over 100 cycles varying  $\mu$ , the results of which are shown in Figure 3.  $\mu$  is varied



**Figure 2.** Samples of time series of  $G$ ,  $M$  and  $\mu$  in three cases with  $\mu(0)$  equal to 24.5, 27 and 29.5 respectively. The values of  $M$  are here multiplied by 800 and those of  $\mu$  are divided by 80 so as to obtain a clearer comparison. In particular, a negative correlation between  $G$  and  $M$ , as well as a positive correlation between  $M$  and  $\mu$  are clearly visible.



**Figure 3.** Part (a): Correlation between the total income  $\mu$  and the Gini index  $G$ . Part (b): Correlation between the mobility  $M$  and the Gini index  $G$ .



**Figure 4.** Correlation between the total income  $\mu$  and the mobility  $M$ .

$\mu(0)$	22.0	24.5	27.0	29.5	32.0
$R_{M\mu}$	$0.951 \pm 0.007$	$0.950 \pm 0.006$	$0.960 \pm 0.006$	$0.972 \pm 0.005$	$0.981 \pm 0.004$

**Table 3.** Correlations  $R_{M\mu}$  (mobility index and total income) computed in five cases with different values of the initial total income  $\mu$ . Again, noise amplitude  $\Gamma$  is equal to 0.001 and averages are taken out of 50 realizations, each of 5000 integration steps.

approximately between  $21 < \mu < 28$ , corresponding for  $G$  to  $0.36 < G < 0.41$ . Each simulation consists of 50 stochastic realizations, each over 5000 steps and starting from the same equilibrium configuration; the solid circles in the plot represent the simulation data.

Figure 3a shows that the  $M$ - $G$  correlation is positive in the interval  $0.36 < G < 0.38$ ; for  $G > 0.38$ , the aforementioned correlation becomes negative. Therefore, according to our model, the “Great Gatsby law”, which states that the correlation between inequality and economic mobility is negative, strictly holds for  $G > 0.38$ . This is actually a range representing the pre-taxation values of  $G$  that includes most industrialized countries.

Figure 3b shows that the  $\mu$ - $G$  correlation is positive in the interval  $0.36 < G < 0.395$  but gets negative thereafter. We thus identify a window of values for  $G$  for which the influx of wealth to the system contributes in decreasing inequality.

Finally, Figure 4 shows that the total income  $\mu$  and mobility  $M$  always have a strong positive correlation which shows a slow increase with increasing  $G$ . This could be understood from an established thermodynamic allusion: in a canonical ensemble, for any reasonable definition of mobility, we expect a strong positive correlation between mobility and temperature; and in turn temperature variations will be strongly correlated with the variations in the free energy (corresponding to income in our case).

#### 4. Conclusion

In this article, we proposed two different models to analyze the time evolution of income distribution resulting from multiple economic exchanges, in the presence of a multiplicative noise (abiding the Ito formulation). The presence of noise causes a continuous dynamical adjustment of the income distribution, while still staying reasonably close to the large time steady state limit that it would have reached in the absence of noise. Ensemble averaging over a large set of stochastic realizations, we observed the emergence of correlations between the Gini inequality index  $G$  and a suitably defined mobility index  $M$ . The respective correlations between mobility  $M$  with the Gini index  $G$  and that between total income  $\mu$  with the Gini index  $G$ , for the time varying case as we consider here, depict association between these quantities. Both the mobility  $M$  and the total income  $\mu$  show steady decrease with increasing  $G$ , a reflection of the fact that an increasing inequality contributes to decreased social mobility (Figure 3). On the other hand, an increasing inequality (reflected by an increasing value of  $G$ ) portrays the strength of interaction between the social mobility and total income

which then shows a marginal steady increase (Figure 4). Some relevant comparisons with results from an equivalent additive noise case are also drawn.

Probably, a more realistic model should involve a weighted combination of both additive and multiplicative stochastic perturbation. Indeed, certain events act as additive noise, whereas others are more properly represented by multiplicative noise. While economics modeling is replete with examples of application of additive noise [19, 20], implementation of multiplicative noise is also not unknown [21, 22, 23].

Correlated noise spectra, like Ornstein-Uhlenbeck, as used in other branches of material science [24], could be considered as well, which is one of our ongoing research projects. More complicated noise structures, resembling power-law scaling have found popular applications in cognition science [25], another possibility for future investigation.

A further extension of the models developed in [11] and here could involve studying the impact of the coefficients  $C_{hk}^i$  themselves changing with the income distribution. Finally, it would be of great interest to investigate the dependence of the entire dynamical process, both on the amplitude as also on the nature of the noise distribution, as alluded to in some of the earlier references in other fields.

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